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LETTER TO THE EDITOR

Automorphisms of the Yang–Baxter equations for the chiral Potts model

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Abstract
For more than a decade now, the chiral Potts model in statistical mechanics has attracted much attention. A number of people have written quite extensively about it. The solutions give rise to a curve over $\mathbb{C}$. Au-Yang and Perk found a large subgroup of the automorphism group of this curve. In this letter, we compute the automorphism group precisely.

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1. Introduction

The chiral Potts model, of statistical mechanics, now has a venerable history. The star–triangle relations (the Yang–Baxter equations) impose constraints on the parameters. For the N-state chiral Potts model, these equations determine a curve in complex projective 3-space $\mathbb{P}^3$, of genus $N^2(N-2)+1$. The geometry of this curve has been one of the mysteries of the subject. Throughout this letter, we will assume $N \geqslant 3$. Then the genus of the chiral Potts curve is $\geqslant 2$, and the automorphism group is finite.

Barry McCoy has been challenging algebraic geometers to use algebraic geometry to tell him something new about the chiral Potts model. He issued a general challenge to algebraic geometers, as well as several more specific ones, to us. There are many of these questions that we do not yet understand. But here is one we can answer.

In [1], Au-Yang and Perk found a group G, of order $4N^3$, acting on this curve. The group has been very useful; the reader is referred particularly to the elegant [2] and [3]. But the problem of determining the full automorphism group has remained. The aim here is to solve this, as follows:

Theorem 1.1. *Let N (the number of states) be an integer $N \geqslant 3$. For every pair $(k, k') \in \mathbb{C}^2$ with $k^2 + \{k'\}^2 = 1$, and $k \neq 0 \neq k'$, there is a Yang–Baxter curve $X = X(k, k')$.*

(i) There is an exact sequence

$$1 \longrightarrow H \longrightarrow \mathrm{Aut}_X \longrightarrow A \longrightarrow 1,$$

where H is a copy of $(\mathbb{Z}/N)^3$ and A is a subgroup of the symmetric group S_4.

(ii) There are exactly 12 points (k, k'), corresponding to the zeros of $(k^2+1)(k^2-2)(2k^2-1)$, for which A is the dihedral group D_8 and Aut_X has order $8N^3$.

(iii) There are exactly eight points (k, k'), corresponding to the zeros of $k^4 - k^2 + 1$, for which A is the alternating group A_4 and Aut_X has order $12N^3$.

(iv) Otherwise A is the 4-group V_4 and Aut_X is the group described by Au-Yang and Perk, of order $4N^3$.

(v) In every case Aut_X is a subgroup of the normalizer of a maximal torus in $PGL_4(\mathbb{C})$ and there is an elliptic curve $E = E(k, k')$ such that $A = \mathrm{Aut}_E/(\pm 1)$.

The proof consists of first finding an upper bound for Aut_X, then a lower bound, by writing down automorphisms, and observing the coincidence of these bounds.

2. An upper bound on the automorphism group

We begin by recalling a crucial result, due to Ciliberto and Lazarsfeld. Recall that a g^r_d on a smooth projective curve C is a linear system of degree d and (projective) dimension r. A g^r_d that is without base-points is identified with a morphism $\phi : C \to \Gamma$, where Γ is a curve in $\mathbb{P}^r$ lying in no hyperplane and $d = \deg \phi . \deg \Gamma$.

Theorem 2.1. *Let X be a smooth curve, which is embedded in $\mathbb{P}^3$ as a complete intersection of two surfaces of degrees $h, k \geqslant 3$. Then the g^3_{hk} defining this embedding is the unique g^3_{hk} on X.*

Proof. For $h, k \geqslant 4$, this is theorem 2.6 in [4]. For $h = 3$, $k \geqslant 3$, see corollary 2.5 loc. cit. Note that in the case $h = k \geqslant 3$, which is the case that interests us here, the inequalities simplify, and the proof of theorem 2.6 of [4] can be seen to also handle the case $h = k = 3$. □

Corollary 2.2. *Let X be a smooth curve, which is embedded in $\mathbb{P}^3$ as a complete intersection of two surfaces of degrees $h, k \geqslant 3$. Any automorphism of X extends to an automorphism of $\mathbb{P}^3$.*

Proof. Any automorphism of X extends to an automorphism of the unique g^3_{hk} of theorem 2.1. □

Next we specialize this to the curve of the Yang–Baxter equations. We remind the reader. The number of states is an integer $N \geqslant 3$. Let k, k' be auxiliary parameters, with $k \neq 0 \neq k'$, so that $k^2 + \{k'\}^2 = 1$. The curve X is the smooth complete intersection determined by the two equations

$$S_1 := a^N + k'b^N - kd^N = 0$$
$$S_2 := k'a^N + b^N - kc^N = 0$$

where $a, \ldots, d$ are homogeneous co-ordinates on $\mathbb{P}^3$. By corollary 2.2, the action of Aut_X on X extends to an action on $\mathbb{P}^3$.

Lemma 2.3. Aut_X *permutes the set of four points* $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ *and* $(0, 0, 0, 1)$.

Proof. Let V denote the vector space of degree N surfaces containing X. Then V is two dimensional. Denote by L its projectivization. Then Aut_X acts on L and L is a copy of $\mathbb{P}^1$.

Now consider Hessians. The relevant facts about the Hessian $H(S)$ of a hypersurface S in $\mathbb{P}^n$ are that $H(S)$ is of degree $n+1$ in the coefficients of S, it is non-zero if S is smooth and it vanishes identically if S is a cone. So the Hessian defines a linear system of degree 4 on L. Casual inspection shows that L contains a smooth member (a Fermat surface), so the Hessian is not identically zero on L. Hence L contains at most 4 cones. However, there are at least 4 cones in L, namely, the two equations given above and their linear combinations

$$S_3 := k'S_1 - S_2 = -k^2b^N + kc^N - kk'd^N$$
$$S_4 := S_1 - k'S_2 = k^2a^N + kk'c^N - kd^N$$

whose vertices are the points mentioned above, and we are done. □

Denote by H the kernel of the action of Aut_X on L, so that there is an exact sequence

$$1 \longrightarrow H \longrightarrow \mathrm{Aut}_X \longrightarrow A \longrightarrow 1$$

where A is a subgroup of Aut_L preserving a 4-tuple, so a subgroup of S_4. The elliptic curve $E = E(k, k')$ is the double cover of $\mathbb{P}^1$ branched at the four points $\left\{0, 1, \infty, \frac{1}{k'^2}\right\}$.

Lemma 2.4. *H is isomorphic to* $(\mathbb{Z}/N)^3$.

Proof. By lemma 2.3, H fixes the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, and so is a quotient of a subgroup M in $GL_4(\mathbb{C})$ consisting of diagonal matrices. It is then elementary to verify that we can take the elements of M to have order N, so that H is identified with the quotient of $(\mathbb{Z}/N)^4$ by the subgroup of scalar matrices. Of course, the element of H represented by (r, s, t, u) acts via $(a, b, c, d) \mapsto (ra, sb, tc, ud)$. □

To get an upper bound on Aut_X we shall bound A. For this, it is enough to enumerate the stabilizers in PGL_2 of the various 4-tuples of points in L. This result is very well known, but for lack of a suitable reference we include a proof. Regard the space of 4-tuples as the projectivization of the vector space U_4 of binary quartics, which, with its PGL_2-action, was analysed extensively in the 19th century. We recall some of these results, which may be found in [5], for example.

Let $g = \sum \binom{4}{i} A_i x^{4-i} y^i \in U_4$. Then the ring of PGL_2-invariants is a polynomial ring $\mathbb{C}[S, T]$, where

$$S = A_0A_4 - 4A_1A_3 + 3A_2^2$$

and

$$T = \det \begin{bmatrix} A_0 & A_1 & A_2 \\ A_1 & A_2 & A_3 \\ A_2 & A_3 & A_4 \end{bmatrix}.$$

It follows from the geometric interpretation of the ring of invariants, due to Mumford [7], that for any $g \in U_4$ with distinct linear factors, the orbit closure $\overline{O(g)}$ in $\mathbb{P}(U_4)$ is a hypersurface and is either of degree 2 and defined by $S = 0$, or of degree 3 and defined by $T = 0$, or of degree 6 and defined by $\lambda S^3 - \mu T^2$ for some λ, μ.

Now fix $g \in U_4$, with distinct linear factors. Regard g as giving a point in $\mathbb{P}(U_4)$ and consider its stabilizer Stab_g in PGL_2. The orbit of such a point is determined by the ratio S^3/T^2. In particular, $S(g) = 0$ if and only if g is equivalent to $x^3y + \sqrt{3}x^2y^2 + xy^3$ and $T(g) = 0$ if and only if g is equivalent to $x^4 + y^4$.

Proposition 2.5. Stab_g *is* A_4 *if* $S(g) = 0$, D_8 *if* $T(g) = 0$ *and* V_4 *otherwise.*

Proof. It is an easy exercise to show that Stab_g always contains V_4. Moreover, $\overline{O(g)}$ has degree $4.3.2/\#(\mathrm{Stab}_g)$; this is the special case $n = 4$ of proposition 1.10 of [6] (a result they attribute to Enriques and Fano). Since Stab_g is a subgroup of S_4, the result follows from the discussion above. □

We need to translate this into statements about the parameters k, k'. Note that $kS_3 = -S_1 + k'S_2$ and $kS_4 = k'S_1 - S_2$, so that if $g \in \mathbb{P}(U_4)$ corresponds to $X = X(k, k')$, then $g = xy(x + k'y)(k'x + y)$. It follows that $6S(g) = 1 - k^2 + k^4$ and $-432T(g) = (k^2 - 2)(k^2 + 1)(2k^2 - 1)$.

Corollary 2.6. *There is a short exact sequence*

$$1 \longrightarrow (\mathbb{Z}/N)^3 \longrightarrow \mathrm{Aut}_X \longrightarrow A \longrightarrow 1$$

where A is a subgroup of A_4 if $k^4 - k^2 + 1 = 0$, of D_8 if $(k^2 - 2)(k^2 + 1)(2k^2 - 1) = 0$ and of V_4 otherwise.

Proof. This is an immediate consequence of the previous results. □

3. A lower bound on the automorphism group

We have already observed an action of H, a copy of $(\mathbb{Z}/N)^3$, on $X = X(k, k')$. Put $\zeta = \zeta_{2N} = \exp(2\pi \mathrm{i}/2N)$.

Lemma 3.1. *The maps $\sigma, \tau : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ defined by*

$$\sigma(a, b, c, d) = (b, a, d, c)$$

and

$$\tau(a, b, c, d) = (d, \zeta c, \zeta^{-1} b, a)$$

are automorphisms of X.

Proof. Easy. □

Consequently, the group G generated by H and σ, τ is a subgroup of Aut_X. It is the Au-Yang–Perk group.

Lemma 3.2. *H is a normal subgroup of G, $G/H \cong V_4$ and G has order $4N^3$.*

Proof. There is a permutation action of G on the four points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, with kernel H. It is clear that the image of this action is V_4. □

To go further we consider automorphisms of the whole family of chiral Potts curves. This means that we take the curve $\mathcal{Y}$ of all points $(k, k') \in \mathbb{C}^2$ that satisfy

$$k^2 + \{k'\}^2 = 1 \qquad k \neq 0 \neq k'$$

and the subvariety $\mathcal{X}$ of $\mathbb{P}^3 \times \mathcal{Y}$ defined by the equations

$$\begin{aligned} S_1 &:= a^N + k'b^N - kd^N = 0 \\ S_2 &:= k'a^N + b^N - kc^N = 0 \\ S_3 &:= d^N - k'c^N - ka^N = 0 \\ S_4 &:= -k'd^N + c^N - kb^N = 0. \end{aligned}$$

Note that $\mathcal{Y}$ is the complement in $\mathbb{P}^1$ of a certain 6-tuple D. There is a natural projection $f : \mathcal{X} \longrightarrow \mathcal{Y}$. The fibre over $(k, k') \in \mathcal{Y}$ is the curve $X(k, k')$. The action of G on each curve

$X(k, k')$ is in fact an equivariant action on $f : \mathcal{X} \longrightarrow \mathcal{Y}$, covering the trivial action on $\mathcal{Y}$. The next thing to do is to extend this to an equivariant action of a certain larger group $\mathbb{G}$. For this, put $\omega = \exp(2\pi \mathrm{i}/4N)$ and define maps $\sigma_1, \sigma_2, \sigma_3 : \mathbb{P}^3 \times \mathcal{Y} \longrightarrow \mathbb{P}^3 \times \mathcal{Y}$ by

$$\sigma_1(a, b, c, d; k, k') = (b, a, \omega^{-1}c, \omega^{-1}d; \mathrm{i}k/k', 1/k')$$
$$\sigma_2(a, b, c, d; k, k') = (\omega a, c, b, \omega d; 1/k, \mathrm{i}k'/k)$$

and

$$\sigma_3(a, b, c, d; k, k') = (\omega a, \omega b, d, c; \mathrm{i}k/k', 1/k').$$

We regard these σ_i also as maps $\mathcal{Y} \longrightarrow \mathcal{Y}$, by forgetting the co-ordinates a, b, c, d.

Lemma 3.3. *The σ_i are equivariant automorphisms of the map $f : \mathcal{X} \longrightarrow \mathcal{Y}$.*

Proof. This is a matter of proving that each σ_i preserves the $\mathcal{O}_{\mathcal{Y}}$-module V generated by $S_1, \ldots, S_4$. This is a routine, if unenlightening, calculation and is left to the reader. (Note that $\mathcal{O}_{\mathcal{Y}}$ is the ring $\mathbb{C}[k, k', 1/k, 1/k']/(k^2 + \{k'\}^2 - 1)$.) □

We define $\mathbb{G}$ to be the group generated by H and σ_i. Note that $\mathrm{Aut}_{\mathcal{Y}}$ is identified with the stabilizer in PGL_2 of D. Since f is $\mathbb{G}$-equivariant, there is a natural homomorphism $\phi : \mathbb{G} \longrightarrow \mathrm{Aut}_{\mathcal{Y}}$; denote the image of $\mathbb{G}$ by K and that of σ_i by s_i.

Lemma 3.4. *The kernel of ϕ contains G.*

Proof. What needs to be proved is that $\mathbb{G}$ contains a group acting trivially on $\mathcal{Y}$, and whose action on the fibres is that of G. There is no problem with the subgroup H. Put

$$\sigma = \sigma_1\sigma_3^{-1} \qquad \text{and} \qquad \tau = \sigma_3^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}.$$

The reader can compute that the above elements of $\mathbb{G}$ act trivially on $\mathcal{Y}$, and agree (modulo H) with the σ and τ of lemma 3.1. □

Lemma 3.5. *There is an inhomogeneous co-ordinate t on $\mathcal{Y}$ given by $t = \frac{1-k'}{k}$ and in terms of t, s_1 and s_2 are the fractional linear transformations represented by the matrices*

$$M_1 = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}$$

and

$$M_2 = \frac{1}{\sqrt{-2}} \begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix}$$

respectively.

Proof. Elementary calculation. □

Corollary 3.6. *K is generated by the images in $PGL_2 = PSL_2$ of the matrices M_i.*

Proof. Immediate. □

Lemma 3.7. *If u, v are homogeneous co-ordinates on $\mathbb{P}^1$ such that $t = u/v$, then the deleted 6-tuple D is defined by the polynomial $h := uv(u^4 - v^4)$.*

Proof. Immediate. □

Lemma 3.8. *K is the octahedral group (that is, the rotation group of a regular octahedron) isomorphic to S_4.*

Proof. Since K is finite, we can identify $\mathbb{P}^1$ with the Riemann sphere in such a way that K is a subgroup of PSU_2, which is isomorphic to $SO_3(\mathbb{R})$, and then the linear factors of h define the vertices of a regular octahedron. From this point of view, s_1 and s_2 appear as rotations though $\pi/2$ about orthogonal axes, and so generate the rotation group of the octahedron. □

4. Completion of the proof of theorem 1.1

For $y = (k, k') \in \mathcal{Y}$, let K_y, respectively $\mathbb{G}_y$, denote the stabilizer of y in K, respectively $\mathbb{G}$. Then there is a homomorphism $\phi_y : \mathbb{G}_y \longrightarrow K_y$ whose kernel contains G. By construction, $\mathbb{G}_y$ is a subgroup of Aut_X, when $X = X(k, k')$. Now K_y is cyclic of order 4, 3, 2, 1 according to whether y is a vertex, a face-center, an edge-center, or none of these, on the octahedron. Now the vertices of the octahedron correspond to points (k, k') where the chiral Potts curve is degenerate (that is, singular). Hence $\mathbb{G}_y$ is respectively of order $12N^3$, $8N^3$, $4N^3$, and we deduce that $\mathrm{Aut}_X = \mathbb{G}_y$.

The identification of the quotient group A with $\mathrm{Aut}_E/(\pm 1)$ for some elliptic curve E follows from the fact that A is the stabilizer in PGL_2 of a 4-tuple of distinct points in $\mathbb{P}^1$. We take E to be the double cover of $\mathbb{P}^1$ ramified in these four points, so that E can be given, in affine co-ordinates, by the equation $y^2 = x(x-1)(x-\{k'\}^{-2})$.

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